ON TWO-CONTACT PROBLEMS FOR AN ELASTIC SPHERE

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In the paper the solution to two contact problems for the elastic sphere is adduced. These problems, apparently, are considered for the first time.

1. **The impression of two rigid stamps on an elastic sphere.** We examine the problem of the impression of two rigid symmetrically arranged stamps on an elastic sphere (Fig.1). We assume that the surface of the sphere exterior to the stamp is free from stress and beneath the stamps tangential stresses are absent.

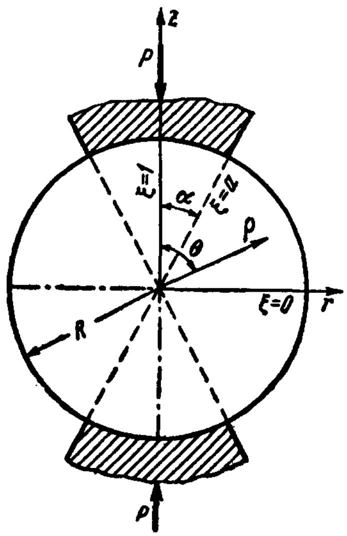

Fig. 1

By virtue of symmetry it is sufficient to consider only deformation for one quarter part of the sphere.

Boundary conditions for the given problem in spherical coordinates are expressed by the relation

$$u_\rho\,|_{\rho=R} = g^*(\theta) \qquad (0 \leqslant \theta < \alpha)$$
$$\tau_{\rho\theta}\,|_{\rho=R} = 0 \qquad (0 \leqslant \theta \leqslant {}^1/_2\,\pi) \qquad (1.1)$$
$$\sigma_\rho\,|_{\rho=R} = 0 \qquad (\alpha < \theta \leqslant {}^1/_2\,\pi)$$

From the condition of symmetry with respect to the z-axis and plane $\theta = \pi/2$ ($z = 0$) we have

$$u_\theta\,|_{\theta=0} = \tau_{\rho\theta}\,|_{\theta=0} = 0$$
$$u_\theta\,|_{\theta={}^1/_2\pi} = \tau_{\rho\theta}\,|_{\theta={}^1/_2\pi} = 0 \qquad (1.2)$$

Here u_ρ and u_θ are radial and meridional components of displacement, and $\tau_{\rho\theta}$ and σ_ρ are the corresponding tangential and normal stresses, $g^*(\theta)$ is a smooth function, determining the form of the surfaces of the stamps and α is a parameter determining the dimensions of the stamps.

The equations of equilibrium in spherical coordinates for the condition of axial symmetry have the form

$$(\lambda + 2\mu)\sin\theta\frac{\partial\Delta}{\partial\theta} + \mu\frac{\partial}{\partial\rho}(2\rho\omega_\varphi\sin\theta) = 0$$
$$(\lambda + 2\mu)\sin\theta\frac{\partial\Delta}{\partial\rho} - \mu\frac{\partial}{\partial\theta}(2\rho\omega_\varphi\sin\theta) = 0 \qquad (1.3)$$

where

$$\Delta = \frac{1}{\rho^2\sin\theta}\left[\frac{\partial}{\partial\rho}(\rho^2 u_\rho\sin\theta) + \frac{\partial}{\partial\theta}(\rho u_\theta\sin\theta)\right], \qquad \omega_\varphi = \frac{1}{2\rho}\left[\frac{\partial}{\partial\rho}(\rho u_\theta) - \frac{\partial u_\rho}{\partial\theta}\right] \qquad (1.4)$$

(λ and μ are Lamé coefficients)

Transforming from the θ coordinate to the coordinate $\xi = \cos\theta$, we shall seek a solution to Equations (1.3) in the form of series (1.5)

$$u_\rho = f(\rho) + \sum_{k=2,4,\ldots}^{\infty} f_k(\rho)P_k(\xi), \qquad u_\theta = \sqrt{1-\xi^2}\sum_{k=2,4,\ldots}^{\infty}\varphi_k(\rho)P_k'(\xi)$$

Here $P_k(\xi)$ are Legendre polynomials, $f_0(\rho)$, $f_k(\rho)$ and $\varphi_k(\rho)$ are unknown functions subject to determination, prime denotes differentiation with respect to ξ.

Substituting Equations (1.5) and (1.4) into Equations (1.3) for the determination of functions $f_0(\rho)$, $f_k(\rho)$ and $\varphi_k(\rho)$, we obtain differential equations of the Euler type, solutions of which we take in the form

$$f_0(\rho) = A_0\frac{\rho}{R}$$
$$f_k(\rho) = -kA_k\left(\frac{\rho}{R}\right)^{k-1} - \frac{\lambda k + \mu(k-2)}{\lambda(k+3)+\mu(k+5)}(k+1)C_k\left(\frac{\rho}{R}\right)^{k+1} \qquad (1.6)$$
$$\varphi_k(\rho) = A_k\left(\frac{\rho}{R}\right)^{k-1} + C_k\left(\frac{\rho}{R}\right)^{k+1} \qquad (k = 2, 4, \ldots)$$

Here the constants of integration A_0, A_k and C_k are determined from the boundary conditions.

In the coordinate system ρ, ξ, φ boundary conditions (1.1) and the condition of symmetry (1.2) take the form

$$u_\rho|_{\rho=R} = g(\xi) = g^*(\theta) \qquad (\cos\alpha = a < \xi \leqslant 1)$$
$$\tau_{\rho\theta}|_{\rho=R} = 0 \quad (0 \leqslant \xi \leqslant 1), \qquad \sigma_\rho|_{\rho=R} = 0 \quad (0 \leqslant \xi < a) \qquad (1.7)$$

$$u_\theta|_{\xi=1} = \tau_{\rho\theta}|_{\xi=1} = 0, \quad u_\theta|_{\xi=0} = \tau_{\rho\theta}|_{\xi=0} = 0 \qquad (0 \leqslant \rho \leqslant R) \qquad (1.8)$$

Employing the usual formulas for stress (1.9)

$$\sigma_\rho = \frac{\lambda}{\rho^2}\left[\frac{\partial}{\partial\rho}(\rho^2 u_\rho) - \frac{\partial}{\partial\xi}(\rho u_\theta\sin\theta)\right] + 2\mu\frac{\partial u_\rho}{\partial\rho}, \qquad \tau_{\rho\theta} = \mu\left[\frac{\partial u_\theta}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial u_\rho}{\partial\xi} - \frac{u_\theta}{\rho}\right]$$

and also relationships (1.5) and (1.6) and considering that $P_{2s}'(0) = 0$, it is easy to show that conditions (1.8) are identically satisfied.

Satisfying further the second of conditions (1.7) we have

$$C_k = -\frac{(k-1)[\lambda(k+3)+\mu(k+5)]}{\lambda k(k+2)+\mu(k^2+2k-1)} A_k \qquad (k=2,4,\ldots) \tag{1.10}$$

and for the determination of coefficients A_0 and A_k $(k = 2, 4, \ldots)$ from the remaining conditions of (1.7) we obtain the following series equations

$$\sum_{k=0,2,\ldots}^{\infty} B_k\left[1+O_1\left(\frac{1}{k}\right)\right]P_k(\xi) = g(\xi) \qquad (a<\xi\leqslant 1)$$
$$\sum_{k=0,2,\ldots}^{\infty} B_k\left(k+\frac{1}{2}\right)P_k(\xi) = 0 \qquad (0\leqslant \xi < a) \tag{1.11}$$

Here (1.12)

$$A_0 = \frac{\mu(\lambda+\mu)B_0}{(3\lambda+2\mu)(\lambda+2\mu)}, \qquad A_k = -\frac{(2k+1)(\lambda+\mu)[\lambda k(k+2)+\mu(k^2+2k-1)]B_k}{2(k-1)(\lambda+2\mu)[\lambda(2k^2+4k+3)+2\mu(k^2+k+1)]}$$

$$O_1\left(\frac{1}{k}\right) = \frac{4\mu^2k^2-(\lambda-\mu)(3\lambda+2\mu)k+2(3\lambda^2+7\lambda\mu+3\mu^2)}{2(\lambda+2\mu)(k-1)[\lambda(2k^2+4k+3)+2\mu(k^2+k+1)]} \tag{1.13}$$

Thus the solution of the formulated problem has reduced itself to the determination of coefficients B_k $(k = 0, 2, \ldots)$ from the "dual" series equations (1.11) containing Legendre polynomials. After determining B_k all constants of integration are determined by relations (1.10) and (1.12).

2. Investigation of "dual" series (1.11). As a preliminary we note that if in the problem analyzed in Section 1 there are present normal loads on the surface of the sphere exterior to the stamp, or if the sphere has an axially symmetric stationary temperature field, then for the solution of these problems instead of system of equations (1.11) we obtain the "dual" series equations in the more general form

$$\sum_{n=0,2,\ldots}^{\infty} B_n\left[1+O\left(\frac{1}{n}\right)\right]P_n(\xi) = g(\xi) \qquad (a<\xi\leqslant 1)$$
$$\sum_{n=0,2,\ldots}^{\infty} B_n\left(n+\frac{1}{2}\right)P_n(\xi) = f(\xi) \qquad (0\leqslant \xi < a) \tag{2.1}$$

Here B_n are unknown coefficients of magnitude $O(n^{-1})$ at $n \to \infty$, tending to zero as n^{-1}; the continuous function $g(\xi)$ has a sectionally continuous derivative; the function $f(\xi)$ is sectionally continuous. Therefore, instead of system (1.11) we consider more general system (2.1).

As a preliminary we consider the system

$$\sum_{n=0}^{\infty} X_n [1 + O(\tfrac{1}{n})] P_n(\xi) = g(\xi) \qquad (0 < a < \xi \leqslant 1)$$
$$\sum_{n=0}^{\infty} X_n (n + \tfrac{1}{2}) P_n(\xi) = f_1(\xi) \qquad (-1 \leqslant \xi < a) \tag{2.2}$$

It is similar to the system of the special case examined by Minkov [1]; the general solution to such systems for $O(n^{-1}) = 0$ was given by Babloian [2].

We proceed to study the "dual" series equations (2.2). For these we designate the value of the second series of (2.2) in the region $a < \xi \leqslant 1$ by

$$V(\xi) = \sum_{n=0}^{\infty} X_n (n + \tfrac{1}{2}) P_n(\xi) \qquad (a < \xi \leqslant 1) \tag{2.3}$$

Then from the second equation of system (2.2) and (2.3) we get for the determination of coefficients X_n

$$X_n = \int_{-1}^{a} f_1(\xi) P_n(\xi)\, d\xi + \int_{a}^{1} V(\xi) P_n(\xi)\, d\xi \tag{2.4}$$

With the help of the well-known formula

$$\cos\left[\left(n + \frac{1}{2}\right) \cos^{-1}\gamma\right] = -\frac{\sqrt{1-\gamma^2}}{\sqrt{2}} \frac{d}{d\gamma} \int_{\gamma}^{1} \frac{P_n(\xi)\, d\xi}{(\xi - \gamma)^{1/2}} \tag{2.5}$$

we transform the first equation of the system (2.2) to the form

$$\sum_{n=0}^{\infty} X_n\left[1 + O\left(\frac{1}{n}\right)\right] \cos\left[\left(n + \frac{1}{2}\right) \cos^{-1}\gamma\right] = -\frac{\sqrt{1-\gamma^2}}{\sqrt{2}} \frac{d}{d\gamma} \int_{\gamma}^{1} \frac{g(\xi)\, d\xi}{(\xi - \gamma)^{1/2}} \tag{2.6}$$
$$(a < \gamma \leqslant 1)$$

Putting the expression for X_n from (2.4) in (2.6) we find

$$\int_{a}^{\gamma} \frac{V(\xi)\, d\xi}{(\gamma - \xi)^{1/2}} = -\int_{a}^{1} V(\xi) S(\xi, \gamma)\, d\xi - \sqrt{1-\gamma^2} \frac{d}{d\gamma} \int_{\gamma}^{1} \frac{g(\xi)\, d\xi}{(\xi - \gamma)^{1/2}} -$$
$$- \int_{-1}^{a} f_1(\xi) [(\gamma - \xi)^{-1/2} + S(\xi, \gamma)]\, d\xi \qquad (a < \gamma \leqslant 1) \tag{2.7}$$

Here

$$S(\xi, \gamma) = \sqrt{2} \sum_{n=0}^{\infty} O\left(\frac{1}{n}\right) P_n(\xi) \cos\left[\left(n + \frac{1}{2}\right) \cos^{-1}\gamma\right] \tag{2.8}$$

For the derivation of relation (2.7) the known series was also used [3]

$$\sum_{n=0}^{\infty} P_n(\xi) \cos\left[\left(n + \frac{1}{2}\right) \cos^{-1}\gamma\right] = \begin{cases} [2(\gamma - \xi)]^{-1/2} & (\gamma > \xi) \\ 0 & (\gamma < \xi) \end{cases} \tag{2.9}$$

Making use of formula of inversion for integral equations of the Abel type for the determination of the function $V(\xi)$ from (2.7) we get the following Fredholm integral equation of the second kind:

$$V(z) = -\frac{1}{\pi}\frac{d}{dz}\int_a^z \frac{d\gamma}{(z-\gamma)^{1/2}}\left\{\int_a^1 V(\xi)\, S(\xi,\gamma)\, d\xi + \sqrt{1-\gamma^2}\frac{d}{d\gamma}\int_\gamma^1 \frac{g(\xi)\, d\xi}{(\xi-\gamma)^{1/2}} + \right.$$

$$\left. + \int_{-1}^a f_1(\xi)\left[(\gamma-\xi)^{-1/2} + S(\xi,\gamma)\right] d\xi\right\} \qquad (a < z \leqslant 1) \qquad (2.10)$$

We assume further, that

$$f_1(\xi) = \begin{cases} V(\xi) & (-1 \leqslant \xi < -a) \\ f(\xi) & (-a < \xi < a) \end{cases} \qquad (2.11)$$

and consider the case where the functions $V(\xi)$ and $f(\xi)$ are even. In this case Equation (2.4) and integral equation (2.10) may be represented in the form

$$X_n = [1 + (-1)^n]\left\{\int_0^a f(\xi)\, P_n(\xi)\, d\xi + \int_a^1 V(\xi)\, P_n(\xi)\, d\xi\right\} \qquad (2.12)$$

$$V(z) = -\frac{1}{\pi}\frac{d}{dz}\int_a^z \frac{d\gamma}{(z-\gamma)^{1/2}}\left\{\int_a^1 \left[(\gamma+\xi)^{-1/2} + S_1(\xi,\gamma)\right] V(\xi)\, d\xi + \right. \qquad (2.13)$$

$$\left. + \int_0^a f(\xi)\left[(\gamma-\xi)^{-1/2} + (\gamma+\xi)^{-1/2} + S_1(\xi,\gamma)\right] d\xi + \sqrt{1-\gamma^2}\frac{d}{d\gamma}\int_\gamma^1 \frac{g(\xi)\, d\xi}{(\xi-\gamma)^{1/2}}\right\}$$

where

$$S_1(\xi,\gamma) = S(\xi,\gamma) + S(-\xi,\gamma) \qquad (2.14)$$

Hence it is seen that the solution (2.12) of system (2.2) coincides with the solution of system (2.1), i.e.

$$X_{2k} = B_{2k}, \qquad X_{2k+1} = 0 \qquad (k = 0, 1, 2, \ldots) \qquad (2.15)$$

and the function $V(z)$, which is determined from integral equation (2.13), will be the value of the second series of system (2.1) in the region $a < z < 1$, i.e. the value of the normal stress beneath the stamps.

For the problem considered in Section 1, in Formulas (2.12) and (2.13), we put $f(\xi) = 0$; then we obtain the following expression for the coefficients B_k of the "dual" series (1.11)

$$B_k = 2\int_a^1 V(\xi)\, P_n(\xi)\, d\xi \qquad (2.16)$$

where the function $V(\xi)$ is obtained from the integral equation

$$V(z) = \int_a^1 K(\xi, z)\, V(\xi)\, d\xi + F(z) \qquad (2.17)$$

Here $K(\xi, z)$ and $F(z)$ are given by the relations

$$K(\xi, z) = -\frac{1}{\pi}\frac{d}{dz}\int_a^z (z-\gamma)^{-1/2}[(\gamma+\xi)^{-1/2} + S_1(\xi,\gamma)]\,d\gamma \qquad (2.18)$$

$$F(z) = -\frac{1}{\pi}\frac{d}{dz}\int_a^z \left(\frac{1-\gamma^2}{z-\gamma}\right)^{1/2} d\gamma \frac{d}{d\gamma}\int_\gamma^1 \frac{g(\xi)\,d\xi}{(\xi-\gamma)^{1/2}} \qquad (2.19)$$

$$S_1(\xi,\gamma) = 2\sqrt{2}\sum_{n=0,2,\ldots}^{\infty} O_1\left(\frac{1}{n}\right) P_n(\xi)\cos\left[\left(n+\frac{1}{2}\right)\ \cos^{-1}\gamma\right] \qquad (2.20)$$

It is easily seen that in the quardant $0 < a < \xi$, $z < 1$, the kernel $K(\xi, z)$ is integrable and Equation (2.17) has a unique solution.

It should be noted that if in (2.10) and (2.11) one takes both functions $V(\xi)$ and $f(\xi)$ odd (this may take place for solutions of other problems of equilibrium of a sphere: as for example, for a problem of equatorial torsion by twisting load on an elastic sphere, clamped by two rigid stamps), then Equation (2.4) and integral equation (2.10) now reduce to the form

$$X_n = [1-(-1)^n]\left(\int_0^a f(\xi)P_n(\xi)\,d\xi + \int_a^1 V(\xi)P_n(\xi)\,d\xi\right) \qquad (2.21)$$

$$V(z) = -\frac{1}{\pi}\frac{d}{dz}\int_a^z \frac{d\gamma}{(z-\gamma)^{1/2}}\left(\int_a^1 [S_2(\xi,\gamma) - (\gamma+\xi)^{-1/2}]V(\xi)\,d\xi + \right. \qquad (2.22)$$

$$\left. + \int_0^a f(\xi)[(\gamma-\xi)^{-1/2} - (\gamma+\xi)^{-1/2} + S_2(\xi,\gamma)]\,d\xi + \sqrt{1-\gamma^2}\frac{d}{d\gamma}\int_\gamma^1 \frac{g(\xi)\,d\xi}{(\xi-\gamma)^{1/2}}\right)$$

where

$$S_2(\xi,\gamma) = S(\xi,\gamma) - S(-\xi,\gamma) \qquad (2.23)$$

In this case the solution to system (2.2) coincides with the solution of the following "dual" series equations

$$\sum_{n=1,3,\ldots}^{\infty} A_n\left[1 + O\left(\frac{1}{n}\right)\right] P_n(\xi) = g(\xi) \qquad (a < \xi \leqslant 1)$$

$$\sum_{n=1,3,\ldots}^{\infty} A_n\left(n+\tfrac{1}{2}\right) P_n(\xi) = f(\xi) \qquad (0 \leqslant \xi < a) \qquad (2.24)$$

i.e.

$$X_{2k+1} = A_{2k+1}, \qquad X_{2k} = 0 \qquad (k = 0, 1, 2, \ldots) \qquad (2.25)$$

and the function $V(z)$ entering in integral equation (2.22) will also be the value of the second series of system (2.24) in the region $a < z < 1$.

In exactly the same way the solution of "dual" series equations (2.1) and (2.24) may be reduced to the determination of function $W(z)$ from integral equations of the type (2.13) and (2.22), where $W(z)$ is the value of the first series of system (2.1) and (2.24) in the region $0 < z < a$.

In concluding this section we note that for the analysis of the state of stress of an elastic sphere impressed by two identical rigid stamps, we separately investigate the case of symmetric and antisymmetric stresses on the elastic sphere. In the first place this makes more clear the formulas and "dual" series equations obtained, and, secondly, each of these stresses has independent significance, since they correspond to the definite character of the sphere. It follows therefore that the case of arbitrary stresses on an elastic sphere due to two identical rigid stamps, symmetrically placed, may be obtained from (2.12) and (2.21) by means of the superposition of two above indicated cases.

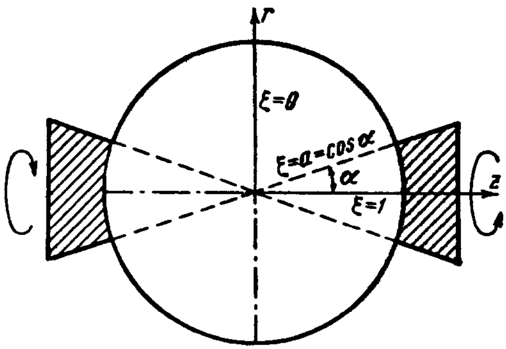

Fig. 2

3. Torsion of an elastic sphere by the action of two stamps, rigidly connected to the sphere. Analogously one may solve the problem of torsion of a continuous elastic sphere when it is twisted by two identical rigid stamps connected to the sphere.

The remaining part of the surface of the sphere, for simplicity, is considered free from external loads (Fig.2).

In a spherical system of coordinates (t, ξ, φ) this problem reduces to the integration of Equation [4 and 5]

$$\frac{\partial^2 \Psi}{\partial t^2} + (1-\xi^2)\frac{\partial^2 \Psi}{\partial \xi^2} + 3\frac{\partial \Psi}{\partial t} - 4\xi\frac{\partial \Psi}{\partial \xi} = 0 \tag{3.1}$$

where $\Psi(t, \xi)$ is a function of the displacement.

From this the stresses $\tau_{\xi\varphi}$ and $\tau_{t\varphi}$ and displacement v are expressed by the function $\Psi(t, \xi)$ by Formulas

$$\tau_{\xi\varphi} = G(1-\xi^2)\frac{\partial \Psi}{\partial \xi}, \qquad \tau_{t\varphi} = G\sqrt{1-\xi^2}\frac{\partial \Psi}{\partial t}, \qquad v = Re^t\sqrt{1-\xi^2}\,\Psi(t, \xi) \tag{3.2}$$

Boundary conditions and the condition of symmetry for the considered problem will have the form

$$\begin{aligned} \tau_{t\varphi}(0, \xi) &= 0 \qquad && (0 \leqslant \xi < a) \\ v(0, \xi) &= R\beta\sqrt{1-\xi^2} \qquad && (a < \xi \leqslant 1) \\ v(t, 0) &= 0 \qquad && (-\infty < t \leqslant 0) \end{aligned} \tag{3.3}$$

where β is angle of twist of the stamps and $R\sqrt{1-\xi^2} = r$ is the distance of points on the surface of the sphere from the z-axis. We seek the function $\Psi(t, \xi)$ in the form of a series

$$\Psi(t, \xi) = \sum_{n=2,4,\ldots}^{\infty} Y_n e^{(n-1)t} P_n'(\xi) \tag{3.4}$$

It is easily seen that the last condition of (3.3) is satisfied identically. To satisfy the remaining conditions of (3.3) for the determination of the unknown coefficients Y_n, appearing in (3.4), we obtain "dual" series equations

$$\sum_{n=2,\,4,\ldots}^{\infty} X_n\left(1+\frac{1.5}{n-1}\right)P_n'(\xi)=\beta \qquad (a<\xi\leqslant 1)$$
$$\sum_{n=2,\,4,\ldots}^{\infty} X_n\left(n+\tfrac{1}{2}\right)P_n'(\xi)=0 \qquad (0\leqslant\xi<a) \tag{3.5}$$

Here we introduce the notation

$$(n-1)\,Y_n=\left(n+\tfrac{1}{2}\right)X_n \tag{3.6}$$

As a preliminary we examine the system

$$\sum_{n=0,\,2,\ldots}^{\infty} X_n\left(1+\frac{1,5}{n-1}\right)P_n(\xi)=\beta\xi+c_0 \qquad (a<\xi\leqslant 1)$$
$$\sum_{n=0,\,2,\ldots}^{\infty} X_n\left(n+\tfrac{1}{2}\right)P_n(\xi)=c \qquad (0\leqslant\xi<a) \tag{3.7}$$

where c_0 and c are as yet unknown constants.

From (3.5) and (3.7) it is seen that one of the constants of value c_0 or c may be selected arbitrarily (for example $c = 0$) and the other constant is determined in the following manner: the solution of system (3.7) (as in system (2.1)) reduces to the determination of a function $V(z)$ from integral equation (2.13). If the solution of this equation is sought in the form of a power series $(z - a)$, then it is easily shown that this series must have the form

$$V(z)=\sum_{n=0}^{\infty} a_n\,(z-a)^{\frac{n-1}{2}} \tag{3.8}$$

Since the function $V'(z)$ represents the tangential stress $\tau_{t\varphi}(0,z)$ in the region $a<z\leqslant 1$ (beneath the stamp), then in this region it must be summable, i.e.

$$\int_a^1 V'(z)\,dz=V(1)-V(a)<\infty \tag{3.9}$$

This relation occurs only in the case if

$$a_0=0 \tag{3.10}$$

The value of constant c_0 is determined from condition (3.10).

We note that systems (3.5) and (3.7) are equivalent only for condition (3.10).

Thus by virtue of (2.1), (2.12) to (2.14) and (3.7) the analysis of the problem above on torsion of an elastic sphere by two rigid stamps reduces to the solution of a Fredholm integral equation of the second kind

$$V(z) = \int_a^1 K(\xi, z) V(\xi)\, d\xi + F_1(z) \qquad (3.11)$$

where

$$K(\xi, z) = -\frac{1}{\pi}\frac{d}{dz}\int_a^z \frac{(\gamma+\xi)^{-1/2} + S_1(\xi, \gamma)}{(z-\gamma)^{1/2}}\, d\gamma \qquad (3.12)$$

$$F_1(z) = \frac{c_0 + \beta(3z-1-a)}{\pi}\left(\frac{1+a}{z-a}\right)^{1/2} + \frac{3\beta z + c_0}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1+a}{z-a}\right)^{1/2}\right] \qquad (3.13)$$

$$S_1(\xi, \gamma) = 3\sqrt{2}\sum_{n=0,2,\ldots}^{\infty}\frac{P_n(\xi)}{n-1}\cos\left[\left(n+\frac{1}{2}\right)\cos^{-1}\gamma\right] \qquad (3.14)$$

After certain transformations Expression (3.13) may be also written in the form

$$S_1(\xi, \gamma) = -3\left(\gamma\sqrt{\xi+\gamma} + \sqrt{1-\gamma^2}\sqrt{\xi-\gamma}\right) - \frac{3\xi}{2}\left[(2\gamma+1)\sqrt{1-\gamma}\cos^{-1}\frac{\sqrt{1-\gamma^2}-\sqrt{\xi^2-\gamma^2}}{1+\xi} - (2\gamma-1)\sqrt{1+\gamma}\ln\frac{\sqrt{1+\gamma}+\sqrt{\xi+\gamma}}{\sqrt{1-\gamma}+\sqrt{\xi-\gamma}}\right] \quad \text{for } \gamma<\xi$$

$$S_1(\xi, \gamma) = -3\gamma\left(\sqrt{\gamma+\xi} + \sqrt{\gamma-\xi}\right) - \frac{3\xi}{2}\left[(2\gamma+1)\sqrt{1-\gamma}\times \cos^{-1}\frac{1-\gamma+\sqrt{\gamma^2-\xi^2}}{\sqrt{1-\xi^2}} - (2\gamma-1)\sqrt{1+\gamma}\ln\frac{\sqrt{1+\gamma}+\sqrt{\gamma+\xi}}{\sqrt{1+\gamma}+\sqrt{\gamma-\xi}}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\right] \quad \text{for } \gamma>\xi \qquad (3.15)$$

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